ELLIPSE ROTATION UNDER A PRESSURE

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Abstract
The paper deals with the turning of an ellipse or an ellipsoid in a situation when a normal force affects on it from opposite sides. We assume that the friction with the contact areas is shear and the parameters of the ellipse are known. Potential applications of this task are very varied. For example, it can be the orientation of an elliptical sample by covering and underlying glass of a microscope, the orientation of pebbles or beans in the press, or a simple measurement of the diameter of a wire with a sliding scale.

Keywords: *ellipse; ellipsoid; orientation; normal force.*

INTRODUCTION
The authors of this article have run into the problem of determining transverse dimensions of the sample of nearly elliptical shape (in space this is an ellipsoid, for which the derived formulas are identical) several times. In practice, it is often the subject of a cylindrical shape with a significant ellipticity, which is exposed to effects of parallel surfaces during the measurement. In this article, we focus on the situation when the device determines distance of these parallel areas adjacent to the measured ellipse.
A similar task would come in case of optical microscope, where the orientation of ellipsoid is affected by the force of planparallel surfaces of covering and underlying glass and the dimension is set by a photograph orthogonal to their flat surface. Taking into consideration our assumption that there is a shear friction between an ellipsoid and planparallel surfaces, the angle of rotation will be given by the curvature of ellipse surface as well as the coefficient of shear friction μ.
The aim of this study is to find an equation for the angle formed by the main halfaxis of the subject of the elliptical shape with planparallel planes, which compress this subject, and that with the coefficient of shear friction between the planes and the subject being known.

MATERIALS AND METHODS
To illustrate and demonstrate the application of discovered connections, a situation when we try to set the exact cross dimension of a human hair using micrometre will be presented. For the measurements we used a digital micrometer JIGO-YT12.7 with a resolution of 0.001 mm.
After this measurement, the sample was repeatedly dehydrated in ethanol baths and deluged by resin and after hardening by mikrotom we took a series of transverse sections of the thickness of 15 μm. Their geometry was subsequently determined under an optical microscope. Cut surface was approximated by ellipse, therefore, the dimension of the main and secondary half-axis was identified. When compressing the flat jaws of the micrometer on the hair 30 year old woman it comes to continual orientation of hair ellipse between the jaws of micrometre (fig. 1) until the moment when sliding friction equals the force rotating the hair. Proportion of the main and adjacent half-axis of hair of white man is approximately 1.62 *(Skřontová, et al., 2017)*. Our goal is to set the final angle, or more precisely the angle interval, in which the oriented ellipse is located towards the flat surface orthogonal to planparallel surfaces. If the angle will be smaller than critical angle α_c, there will be no orientation caused by sliding friction. However, if the initial orientation of ellipse will be bigger, it would cause its rotation and the result angle will be α_c. Micrometre measures transverse dimension, which is slightly bigger than the dimension of the smallest dimension of an ellipse and thus the length of its adjacent half-axis.

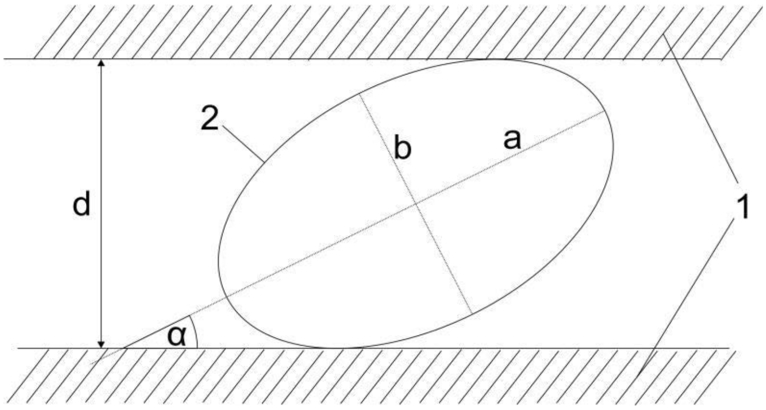

Fig. 1 1-jaws, 2-sample

The aim is to establish the value of this variance on account of the coefficient of shear friction of the pair of materials μ and parameters of ellipse a,b.
Our fundamental assumption is that curvature in a touching point, which means its second derivative in a critical angle must equal shear friction μ. From this attribute we come to the conclusion of this article thanks to the attached method *(Makovický, et al., 2013)*.

RESULTS AND DISCUSSION

Without loss of generality we can transform the task of fig. 1 and solve the easier task (fig. 2).

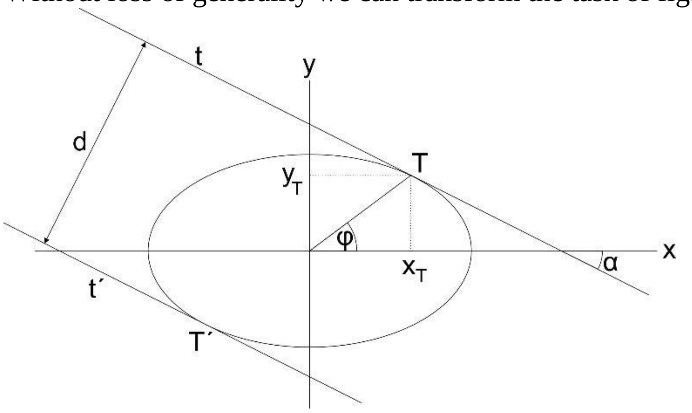

Fig. 2 Ellipse scheme

The sample (ellipse) is placed with its centre at the beginning of the coordinates system and with half-axes parallel with the axes of the system of coordinates *(Mošna, 2014)*. At an angle $\propto \in (0,-\pi/2)$ we create two parallel tangents with tangent points in the 1st and the 3rd quadrant and we denote them by t and t´. The equation of such an ellipse we can write in the usual form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

The slope of the tangent line to this ellipse has evidently the form *(Dvořáková, et al., 2015)*

$$y' = \frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}} \qquad (2)$$

For the tangent point of the line with the slope $tg \propto$ and our ellipse then holds

$$y_T = b\sqrt{1-\frac{x_T^2}{a^2}} \qquad (3)$$

For the slope of the tangent line then the equality is

$$y'(x_T) = tg\ \alpha = \frac{-bx_T}{a^2\sqrt{1-\frac{x_T^2}{a^2}}} \tag{4}$$

From which we get the coordinate x_T of the tangent point

$$x_T = \sqrt{\frac{a^4\ tg^2\propto}{b^2+a^2\ tg^2\propto}} \tag{5}$$

After substituting into the equation (3) we get also y-coordinate of the tangent point T

$$y_T = b\sqrt{\frac{b^2}{b^2+a^2\ tg^2\propto}} \tag{6}$$

For tg φ of the line connecting the beginning of the coordinates and the tangent point T then holds

$$tg\ \varphi = \frac{b^2}{a^2\ tg\alpha} \tag{7}$$

The equation of the tangent line is then

$$x\ tg\alpha\ - y + \frac{b^2 - tg^2\alpha\ a^2}{\sqrt{b^2+a^2\ tg^2\propto}} = 0 \tag{8}$$

For the distance of the tangent from the beginning of the coordinates we get the relationship

$$\frac{d}{2} = cos\alpha\frac{b^2-a^2\ tg^2\alpha}{\sqrt{b^2+a^2\ tg^2\propto}} \tag{9}$$

According to the above assumption, the rotation of the elliptical sample occurs if the second derivative of the surface of the sample is greater than the module of the shear friction

$y'' > \mu.$

$$y'' = \frac{-ba}{(a^2-x^2)^{3/2}} \tag{10}$$

For the limit angle φ_c, in which compensation of the friction and rotational forces occurs we can write for the coordinates of the critical rotation

$$x_c = \sqrt{a^2 - \sqrt[3]{(-\frac{ba}{\mu})^2}} \qquad , y_c = \sqrt{b^2(1-\frac{a^2-\sqrt[3]{(-\frac{ba}{\mu})^2}}{a^2})} \tag{11}$$

For $tg\ \varphi$ is therefore

$$tg\varphi = \frac{\frac{b}{a}\cdot\sqrt[3]{\frac{ab}{\mu}}}{\sqrt{a^2-\sqrt[3]{(-\frac{ba}{\mu})^2}}} \tag{12}$$

Using the equation (7), we get

$$tg^2\alpha = \frac{b^2}{a^2}\cdot\frac{a^2-\sqrt[3]{(-\frac{ba}{\mu})^2}}{\sqrt[3]{(-\frac{ba}{\mu})^2}} \tag{13}$$

And by using the equation (9) we finally get the critical distance d_c .
Our main question is, what is the relationship between the half-axis b and d/2. Let's take the specific values for human hair:

For half-axis a = 47,5 μm, b = 29,3 μm, μ = 0,11 is $\alpha_c = 47{,}6^o$ a $d_c = 41{,}8$ μm *(Jelen, et al., 2014)*. The deviation amounts to 43 %. In case of human hair, the determination of the minimum dimension b of the ellipse with the help of a micrometer is only therefore possible with a tolerance of 43%.

Another comparison we can make based on these measurements, from which we know the cross-sectional dimensions a,b of the female hair identified from the ortogonal cut and at the same time from data measured by a digital micrometer. These measurements then indicate that the maximum measured value actually corresponds with the limit angle specified by equation 13. These measured data served to validate our designated equations, which are the main result of this article.

DISCUSSION

As we have just shown, when using the real values the deviation of the measured value from the length of the minor half-axis of the ellipse may be up to 50 %. Such a value can be significant e.g. in the case that we want to read the diameter of the hair of the horse, where the length of the horse hair we use as a time-recording medium of the condition level of the horse and the diameter of the horse hair in a given location corresponds to the condition of the horse at the time of the growth of this section.
In such cases, in the measurement of soft materials is, however, at the same time necessary to count with the fact that the deformation of the material during a mechanical measurement occurs by the influence of pressure, and it may be even greater than our 43 %.
Another factor influencing the similar measurement is certainly the imperfection of elliptical shape in cross-section of the sample. We consider, however, the elliptical approximation to be an essential second step to the expression of the shape of the sample.
Similar relations corresponding to our relationship 2 are the same result as derived by *Mošna (2014)*.

CONCLUSIONS

Formulas derived above present the base for error estimation while measuring loosely placed samples in a contact way. It is a reflection, which every experimenter facing such problem should take into consideration. It is clear that relatively complicated resulting formulas (12), (13) could be in case of particular samples with a small dispersion of values simplified and some components could be omitted. This could be also done in case of hair, which we have chosen as illustrative. Intentionally we present the formulas in full version, so that their future use would be as general as possible.

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